Two-Sided (Bilateral) Z-Transform Correspondence Table

Bilateral definition: $\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$.

Notes: The ROC (region of convergence) is an annulus $r_1 < |z| < r_2$, determined by the growth/decay of $x[n]$. If the ROC includes the unit circle $|z| = 1$, then the DTFT exists and $X(\mathrm{e}^{\mathrm{j}\omega}) = X(z)\big|_{z=\mathrm{e}^{\mathrm{j}\omega}}$.

A. Common Transform Pairs (bilateral definition)

Sequence $x[n]$	$X(z)$	**ROC (typ.)**	**Notes**
$\delta[n]$	1	all z	Unit sample
$\delta[n-n_0]$	z^{-n_0}	all z	Integer delay $n_0 \in \mathbb{Z}$
$\epsilon[n]$	$\dfrac{1}{1-z^{-1}} = \dfrac{z}{z-1}$	$\|z\| > 1$	Right-sided step
$-\,\epsilon[-n-1]$	$\dfrac{1}{1-z^{-1}} = \dfrac{z}{z-1}$	$\|z\| < 1$	Left-sided step (same form, different ROC)
$a^n\,\epsilon[n]$	$\dfrac{1}{1-az^{-1}} = \dfrac{z}{z-a}$	$\|z\| > \|a\|$	Right-sided exponential
$-\,a^n\,\epsilon[-n-1]$	$\dfrac{1}{1-az^{-1}} = \dfrac{z}{z-a}$	$\|z\| < \|a\|$	Left-sided exponential
$n\,a^n\,\epsilon[n]$	$\dfrac{az}{(z-a)^2}$	$\|z\| > \|a\|$	First moment (right-sided)
$-\,n\,a^n\,\epsilon[-n-1]$	$\dfrac{az}{(z-a)^2}$	$\|z\| < \|a\|$	First moment (left-sided)
$\alpha^{\|n\|},\ 0 < \alpha < 1$	$\dfrac{1-\alpha^2}{1-2\alpha z^{-1}+\alpha^2 z^{-2}}$	$\alpha < \|z\| < \alpha^{-1}$	Two-sided exponential (even)
$\epsilon[n] - \epsilon[n-N]$	$\dfrac{1-z^{-N}}{1-z^{-1}}$	$\|z\| > 1$	Right-sided length-N rectangle
$\mathbf{1}_{\{\|n\|\le M\}}$	$\sum_{n=-M}^{M} z^{-n} = \dfrac{z^{M+1}-z^{-M}}{z-1}$	all $z \neq 0$	Symmetric length $2M{+}1$ rectangle

Definitions.

- $\epsilon[n]$ is the unit step; $\epsilon[-n-1] = 1$ for $n \le -1$, else 0.
- $\mathbf{1}_{\{|n|\le M\}}$ equals 1 for $|n| \le M$, else 0.
- For right-sided exponentials $a^n\epsilon[n]$, the pole is at $z = a$ with ROC $|z| > |a|$; for left-sided $-a^n\epsilon[-n-1]$, the same pole but ROC $|z| < |a|$.

B. Useful Properties (bilateral)

Time domain	**z-domain**	**Comments / ROC**
Linearity: $\mathcal{Z}\{ax + by\}$	$aX + bY$	$a, b \in \mathbb{C}$
Time shift (delay): $x[n - n_0]$	$z^{-n_0} X(z)$	ROC unchanged except possible endpoints $0, \infty$
Time reversal: $x[-n]$	$X(z^{-1})$	ROC maps to reciprocal annulus
Convolution: $(x * y)[n]$	$X(z)\, Y(z)$	Intersection of ROCs
Multiplication: $x[n]y[n]$	$\dfrac{1}{2\pi \mathrm{j}} \oint X(\zeta)\, Y\left(\dfrac{z}{\zeta}\right) \dfrac{d\zeta}{\zeta}$	Frequency-domain convolution (Mellin form)
First moment: $n\, x[n]$	$-\, z\, \dfrac{dX}{dz}$	Differentiation in z
Accumulation: $s[n] = \sum_{k=-\infty}^{n} x[k]$	$\dfrac{1}{1 - z^{-1}}\, X(z)$	ROC must exclude $z = 1$ and include overlap
DTFT link	$X(\mathrm{e}^{\mathrm{j}\omega}) = X(z)\big\vert_{z=\mathrm{e}^{\mathrm{j}\omega}}$	Valid if ROC includes $\lvert z\rvert = 1$
Initial value (if $\infty \in$ ROC)	$x[0] = \lim\limits_{z \to \infty} X(z)$	Right-sided inclusion of ∞
Final value (if $1 \in$ ROC, stable)	$\lim\limits_{n \to \infty} x[n] = \lim\limits_{z \to 1} (1 - z^{-1}) X(z)$	Poles inside unit circle except possibly at $z = 1$